LAMPIRAN

Lampiran 1 Perhitungan Lemma 2.1 untuk Matriks $A + B$

$$A + B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{bmatrix} -\delta_1 \left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k - b - 2\alpha_1 & -\delta_1 (\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k) & \delta_1 \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k + \alpha_2 \\ \delta_1 \left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k & \delta_1 \left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) - (b + d) & -\delta_1 \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k \\ 0 & d & -(b + \alpha_2) \end{bmatrix}$$

dengan

$$\delta_1 = (\beta + \gamma\alpha_1),$$

$$k = \frac{b + \alpha_1}{b + \alpha_2 + d}.$$

Lemma 2.1

i) Untuk $\mathcal{R}_0 > 1$

$$A_1 = -tr(A + B) = -(a_{11} + a_{22} + a_{33})$$

$$a_{11} = -\delta_1 \left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k - b$$

$$= -(\beta + \gamma\alpha_1)\left(1 - \frac{b + d}{\beta + \gamma\alpha_1}\right)^2 k < 0,$$

$$a_{22} = (\beta + \gamma\alpha_1)\left(\frac{b + d}{\beta + \gamma\alpha_1} - \frac{(b + d)\left(1 - \frac{b + d}{\beta + \gamma\alpha_1}\right)k}{\beta + \gamma\alpha_1}\right) - b - d$$

karena $R_{0\gamma} > 1$ maka $a_{22} < 0$,

$$a_{33} = -b - \alpha_2 < 0,$$

$$A_1 = -tr(A + B) = -(a_{11} + a_{22} + a_{33}) > 0.$$

ii) $A_2 = J_1 + J_2 + J_3$,

$$J_1 = a_{11}a_{22} - a_{21}a_{12} = (a_{11} + a_{21})a_{22} - a_{21}(a_{11} + a_{22})$$

$$J_1 = -ba_{22} + (b + d)a_{21} > 0,$$

karena

$$a_{21} = (\beta + \gamma\alpha_1)\left(1 - \frac{b+d}{\beta + \gamma\alpha_1}\right)^2 k > 0$$

$$a_{22} = (\beta + \gamma\alpha_1)\left(\frac{b+d}{\beta+\gamma\alpha_1} - \frac{(b+d)\left(1-\frac{b+d}{\beta+\gamma\alpha_1}\right)k}{\beta+\gamma\alpha_1}\right) - b - d < 0$$

karena $\mathcal{R}_0 > 1$ maka $a_{22} < 0$

$J_2 = a_{22}a_{33} - a_{23}a_{32} > 0$

$$a_{22} = (\beta + \gamma\alpha_1)\left(\frac{b+d}{\beta+\gamma\alpha_1} - \frac{(b+d)\left(1-\frac{b+d}{\beta+\gamma\alpha_1}\right)k}{\beta+\gamma\alpha_1}\right) - b - d$$

karena $\mathcal{R}_0 > 1$maka $a_{22} < 0$,

$a_{33} = -b - \alpha_2 < 0$,

$$a_{23} = -\frac{(b+d)(-\beta - \gamma\alpha_1 + b + d)k}{\beta + \gamma\alpha_1} < 0,$$

$a_{32} = d > 0$,

$J_2 > 0$,

$J_3 = a_{11}a_{33} > 0$,

karena

$$a_{11} = -\delta_1\left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k - b$$

$$= -(\beta + \gamma\alpha_1)\left(1 - \frac{b+d}{\beta+\gamma\alpha_1}\right)^2 k < 0,$$

$a_{33} = -b - \alpha_2 < 0$,

$A_2 = J_1 + J_2 + J_3 > 0$.

iii) $A_3 = -\det(A + B)$

$$= b(b+\alpha_2)\left[(b+d) - (\beta+\gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0}\left(1-\frac{1}{\mathcal{R}_0}\right)k\right)\right] >$$

$$b(b+\alpha_2)\left[(b+d) - (\beta+\gamma\alpha_1)\frac{1}{\mathcal{R}_0}\right] = 0.$$

iv) $A_1A_2 - A_3 = a_{13}a_{21}a_{32} - a_{11}a_{22}a_{33}$

dengan

$a_{11} < 0, a_{22} < 0, a_{33} < 0,$

$$a_{13} = \delta_1 \frac{1}{\mathcal{R}_0}\left(1-\frac{1}{\mathcal{R}_0}\right)k + \alpha_2 > 0,$$

$$a_{21} = (\beta+\gamma\alpha_1)\left(1-\frac{b+d}{\beta+\gamma\alpha_1}\right)^2 k > 0,$$

$a_{32} = d > 0,$

diperoleh

$A_1A_2 - A_3 > 0.$

Lampiran 2 Perhitungan Lemma 2.1 untuk Matriks $A - B$

$$A - B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix},$$

$$= \begin{bmatrix} -\delta_2\left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k - (b + 2\alpha_1) & -\delta_2(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k) & \delta_2 \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k + \alpha_2 \\ \delta_2\left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k & \delta_2\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) - (b + d + 2\alpha_1) & -\delta_2 \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k \\ 0 & d & -(b + 2\alpha_1 + \alpha_2) \end{bmatrix}$$

dengan

$$\delta_2 = (\beta - \gamma\alpha_1),$$

$$k = \frac{b + \alpha_1}{b + \alpha_2 + d}.$$

Lemma 2.1

i) $A_1 = -tr(A - B) = -(b_{11} + b_{22} + b_{33})$

dengan

$$b_{11} = -\delta_2\left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k - (b + 2\alpha_1)$$

$$= -(\beta - \gamma\alpha_1)\left(1 - \frac{b + d}{\beta + \gamma\alpha_1}\right)^2 k - (b + 2\alpha_1) < 0,$$

$$b_{22} = \delta_2\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) - (b + d + 2\alpha_1),\ R_{0\gamma} > 1$$

$b_{22} < 0,$

$b_{33} = -(b + 2\alpha_1 + \alpha_2) < 0$

maka

$A_1 = -tr(A - B) = -(b_{11} + b_{22} + b_{33}) > 0.$

ii) $A_2 = J_1 + J_2 + J_3$

$J_1 = b_{11}b_{22} - b_{12}b_{21}$

dengan

$b_{11}b_{22} < 0,$

$$b_{12}b_{21} = -{\delta_2}^2\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right)\left(1 - \frac{1}{{\mathcal{R}_0}^2}\right)^2 k \le 0,$$

$J_1 = b_{11}b_{22} - b_{12}b_{21} < 0,$

$J_2 = b_{22}b_{33} - b_{23}b_{32}$

$$= \left(-\delta_2\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) + (b + d + 2\alpha_1)\right)$$

$$((b + 2\alpha_1 + \alpha_2)) - d(-\delta_2 \frac{1}{\mathcal{R}_0}(1 - \frac{1}{\mathcal{R}_0})k)$$

jika $\beta - \gamma\alpha_1 \geq 0$

$-\delta_2\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) > 0,$

$(b + d + 2\alpha_1) > 0,$

$(b + 2\alpha_1 + \alpha_2) > 0,$

$-d\left(-\delta_2 \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) > 0,$

$J_2 > 0,$

Jika $\beta - \gamma\alpha_1 < 0$

$-\delta_2\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) > 0$

$(b + d + 2\alpha_1) > 0,$

$(b + 2\alpha_1 + \alpha_2) > 0,$

$-d\left(-\delta_2 \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) < 0$

karena

$b_{22}b_{33} > b_{23}b_{32}$

diperoleh

$J_2 > 0,$

$J_3 = b_{11}b_{33}$

$b_{11} = -\delta_2\left(1 - \frac{1}{{\mathcal{R}_0}^2}\right)^2 k - (b + 2\alpha_1) < 0,$

$b_{33} = -(b + 2\alpha_1 + \alpha_2) < 0,$

maka

$J_3 > 0.$

iii) $A_3 = -\det(A - B)$

$$= (b + 2\alpha_1)((-(\beta - \gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0} - \left(1 - \frac{1}{\mathcal{R}_0}\right)k\right) + (b + d + 2\alpha_1))$$

$$(b+\alpha_2+2\alpha_1)+d(\beta-\gamma\alpha_1)(1-\frac{1}{R_{0\gamma}})k)$$

$A_3=(b+2\alpha_1)\xi$

$A_3\equiv\xi$

$\xi=(-(\beta-\gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0}-\left(1-\frac{1}{\mathcal{R}_0}\right)k\right)+(b+d+2\alpha_1))$

$$(b+\alpha_2+2\alpha_1)+d(\beta-\gamma\alpha_1)(1-\mathcal{R}_0)k)$$

jika $\beta-\gamma\alpha_1\geq 0$

$-(\beta-\gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0}-\left(1-\frac{1}{\mathcal{R}_0}\right)k\right)>0$

$(b+d+2\alpha_1)>0$

$(b+\alpha_2+2\alpha_1)>0$

$d(\beta-\gamma\alpha_1)\left(1-\frac{1}{\mathcal{R}_0}\right)k>0$

diperoleh

$\xi>0, A_3>0,$

jika $\beta-\gamma\alpha_1<0$

$-(\beta-\gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0}-\left(1-\frac{1}{\mathcal{R}_0}\right)k\right)<0$

$(b+d+2\alpha_1)>0$

$(b+\alpha_2+2\alpha_1)>0$

$d(\beta-\gamma\alpha_1)\left(1-\frac{1}{\mathcal{R}_0}\right)k<0$

karena

$\left(-(\beta-\gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0}-\left(1-\frac{1}{\mathcal{R}_0}\right)k\right)+(b+d+2\alpha_1)\right)(b+\alpha_2+2\alpha_1)>$

$d(\beta-\gamma\alpha_1)\left(1-\frac{1}{\mathcal{R}_0}\right)k$

diperoleh

$\xi>0, A_3>0,$

iv) $A_1A_2-A_3=b_{13}b_{21}b_{32}-b_{11}b_{22}b_{33}$

$$= \left((\beta - \gamma\alpha_1)\left(1 - \frac{1}{R_{0\gamma}}\right)^2 k + b + 2\alpha_1 \right)$$

$$\left(b + d + 2\alpha_1 - (\beta - \gamma\alpha_1)\left(\frac{1}{\mathcal{R}_0} - \frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k\right)\right)$$

$$(b + \alpha_2 + 2\alpha_1) + d(\alpha_2 + (\beta - \gamma\alpha_1)\frac{1}{\mathcal{R}_0}\left(1 - \frac{1}{\mathcal{R}_0}\right)k)$$

$$(\beta - \gamma\alpha_1)\left(1 - \frac{1}{\mathcal{R}_0}\right)^2 k > 0$$

jika $\beta - \gamma\alpha_1 \geq 0$

$b_{13} \geq 0, b_{21} \geq 0, b_{32} > 0, b_{11}b_{33} > 0$, dan $b_{22} < 0$, maka

$A_1A_2 - A_3 > 0$,

jika $\beta - \gamma\alpha_1 < 0$

$$A_1A_2 - A_3 \geq (b + \alpha_1)d(b + \alpha_2) + d\alpha_2(\beta - \gamma\alpha_1)(1 - \frac{1}{\mathcal{R}_0})^2 k$$

$$> (b + \alpha_1)d(b + \alpha_2) + d\alpha_2(\beta - \gamma\alpha_1)$$

$$> d\alpha_1\alpha_2 - d\gamma\alpha_1\alpha_2 \geq 0.$$

Lampiran 3 Pembuktian $\frac{\partial S^*}{\partial \gamma} < 0$

$$S^* = \frac{a}{b}\frac{1}{\mathcal{R}_0}, R_{0\gamma} = \frac{(\beta + \gamma\alpha_1)}{b + d}.$$

Turunan dari S^* adalah

$$\frac{\partial S^*}{\partial \gamma} = \frac{\partial}{\partial \gamma}\left(\frac{a}{b}.\frac{1}{\frac{(\beta + \gamma\alpha_1)}{b + d}}\right) = \frac{\partial}{\partial \gamma}\left(\frac{a}{b}.\frac{b + d}{(\beta + \gamma\alpha_1)}\right).$$

Misalkan

$U = a(b + d),$

$V = b(\beta + \gamma\alpha_1).$

$U' = 0,$

$V' = b\alpha_1.$

Sehingga

$$\frac{\partial S^*}{\partial \gamma} = \frac{U'V - UV'}{V^2},$$

$$= \frac{-b\alpha_1(ab + ad)}{V^2}.$$

Karena titik kesetimbangan endemi eksis dan stabil saat $\mathcal{R}_0 > 1$, maka $(\beta + \gamma\alpha_1) > b + d$, sehingga $V^2 > 0$.

Terbukti

$$\frac{\partial S^*}{\partial \gamma} = \frac{-b\alpha_1(ab + ad)}{V^2} < 0.$$

Lampiran 4 pembuktian $\frac{\partial I^*}{\partial \gamma} > 0$

$$I^* = \frac{a(b+\alpha_2)}{b(b+\alpha_2+d)}\left(1-\frac{1}{\mathcal{R}_0}\right), \mathcal{R}_0 = \frac{(\beta+\gamma\alpha_1)}{b+d}.$$

Turunan dari I^* adalah

$$\frac{\partial I^*}{\partial \gamma} = \frac{\partial}{\partial \gamma}\left(\frac{a(b+\alpha_2)\big(\beta+\gamma\alpha_1-(b+d)\big)}{b(b+\alpha_2+d)(\beta+\gamma\alpha_1)}\right),$$

Misalkan

$U = a(b+\alpha_2)\big(\beta+\gamma\alpha_1-(b+d)\big),$

$V = b(b+\alpha_2+d)(\beta+\gamma\alpha_1),$

$U' = \alpha_1(ab+a\alpha_2),$

$V' = \alpha_1\big(b(b+\alpha_2+d)\big).$

Sehingga

$$\frac{\partial I^*}{\partial \gamma} = \frac{U'V-UV'}{V^2},$$

$U'V = \alpha_1(ab+a\alpha_2)\big(b(b+\alpha_2+d)(\beta+\gamma\alpha_1)\big),$

$UV' = \alpha_1\big(b(b+\alpha_2+d)\big)\Big(a(b+\alpha_2)\big(\beta+\gamma\alpha_1-(b+d)\big)\Big),$

$$U'V-UV = \alpha_1(b^4a+2b^3a\alpha_2+2b^3ad+b^2ad^2+b^2a\alpha_2^2$$
$$+bad^2\alpha_2+ba\alpha_2^2d+3b^2a\alpha_2d) > 0,$$

$V^2 > 0,$

Terbukti

$$\frac{\partial I^*}{\partial \gamma} = \frac{U'V-UV'}{V^2} > 0.$$

Lampiran 5 pembuktian $\frac{\partial R^*}{\partial \gamma} > 0$

$$R^* = \frac{ad}{b(b+\alpha_2+d)}\left(1-\frac{1}{R_{0\gamma}}\right), R_{0\gamma} = \frac{(\beta+\gamma\alpha_1)}{b+d}.$$

Turunan dari R_γ^* adalah

$$\frac{\partial R^*}{\partial \gamma} = \frac{\partial}{\partial \gamma}\left(\frac{ad((\beta+\gamma\alpha_1)-(b+d))}{b(b+\alpha_2+d)(\beta+\gamma\alpha_1)}\right),$$

Misalkan

$U = ad((\beta+\gamma\alpha_1)-(b+d)),$

$V = b(b+\alpha_2+d)(\beta+\gamma\alpha_1),$

$U' = ad\alpha_1,$

$V' = \big(b(b+\alpha_2+d)\big)\alpha_1.$

Sehingga

$\frac{\partial R^*}{\partial \gamma} = \frac{U'V-UV'}{V^2},$

$U'V = ad\alpha_1\big(b(b+\alpha_2+d)(\beta+\gamma\alpha_1)\big),$

$UV' = \big(b(b+\alpha_2+d)\big)\alpha_1\Big(ad\big((\beta+\gamma\alpha_1)-(b+d)\big)\Big),$

$$\begin{aligned} U'V-UV = {} & ad\alpha_1 bb(\beta+\gamma\alpha_1 + ad\alpha_1 b\alpha_2(\beta+\gamma\alpha_1) \\ & +ad\alpha_1 bd(\beta+\gamma\alpha_1) \\ & -b^2\alpha_1\big(ad(\beta+\gamma\alpha_1-b-d)\big) \\ & -b\alpha_1\big(ad(\beta+\gamma\alpha_1-b-d)\big)\alpha_2 \\ & -b\alpha_1\big(ad(\beta+\gamma\alpha_1-b-d)\big)b > 0, \end{aligned}$$

$V^2 > 0,$

Terbukti

$$\frac{\partial R^*}{\partial \gamma} = \frac{U'V - UV'}{V^2} > 0.$$